Extending SDDP-style Algorithms for Multistage Stochastic Programming

Dave Morton

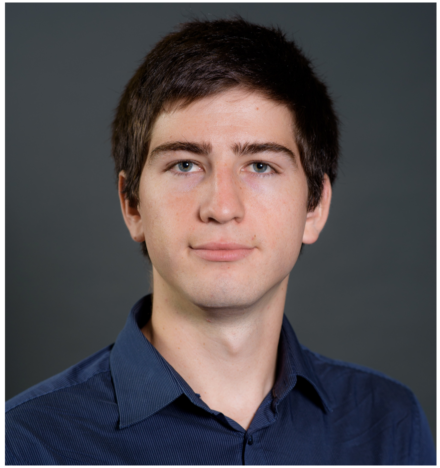
Industrial Engineering & Management Sciences

Northwestern University

Joint work with:

Oscar Dowson, Daniel Duque, and Bernardo Pagnoncelli

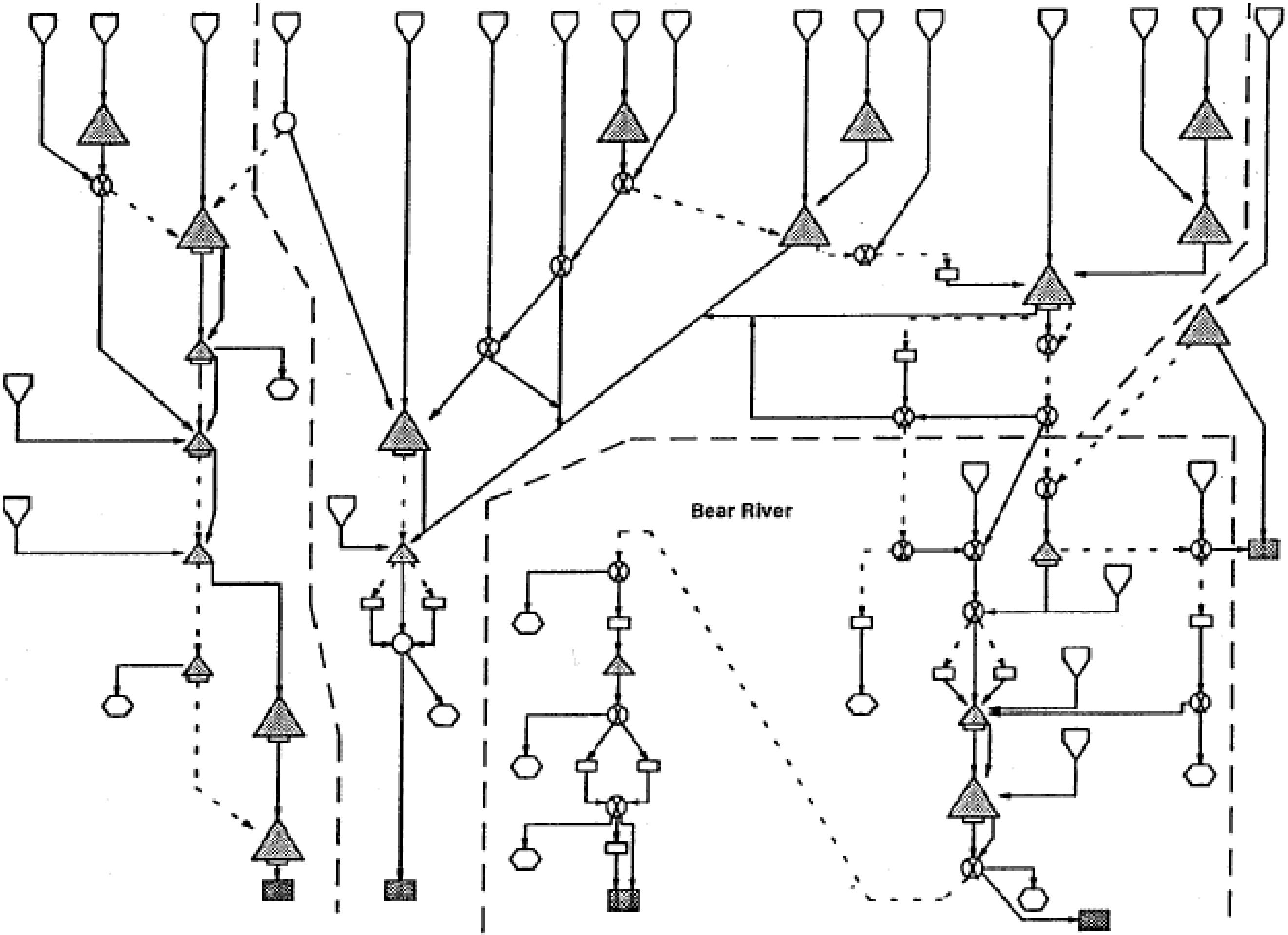
Collaborators



Hydroelectric Power

Itaipu (14 GW)

Yuba, Bear and South Feather Hydrological Basin



SDDP
Stochastic Dual Dynamic Programming

SLP-T

$$
\begin{aligned}
z^* = \min_{x_1 \geq 0} \ & c_1 x_1 + \mathbb{E}_{\xi_2|\xi_1} V_2(x_1, \xi_2) \\
\text{s.t.} \ & A_1 x_1 = B_1 x_0 + b_1
\end{aligned}
$$

where for $t = 2, \ldots, T$,

$$
\begin{aligned}
V_t(x_{t-1}, \xi_t) = \min_{x_t \geq 0} \ & c_t x_t + \mathbb{E}_{\xi_{t+1}|\xi_1,\ldots,\xi_t} V_{t+1}(x_t, \xi_{t+1}) \\
\text{s.t.} \ & A_t x_t = B_t x_{t-1} + b_t
\end{aligned}
$$

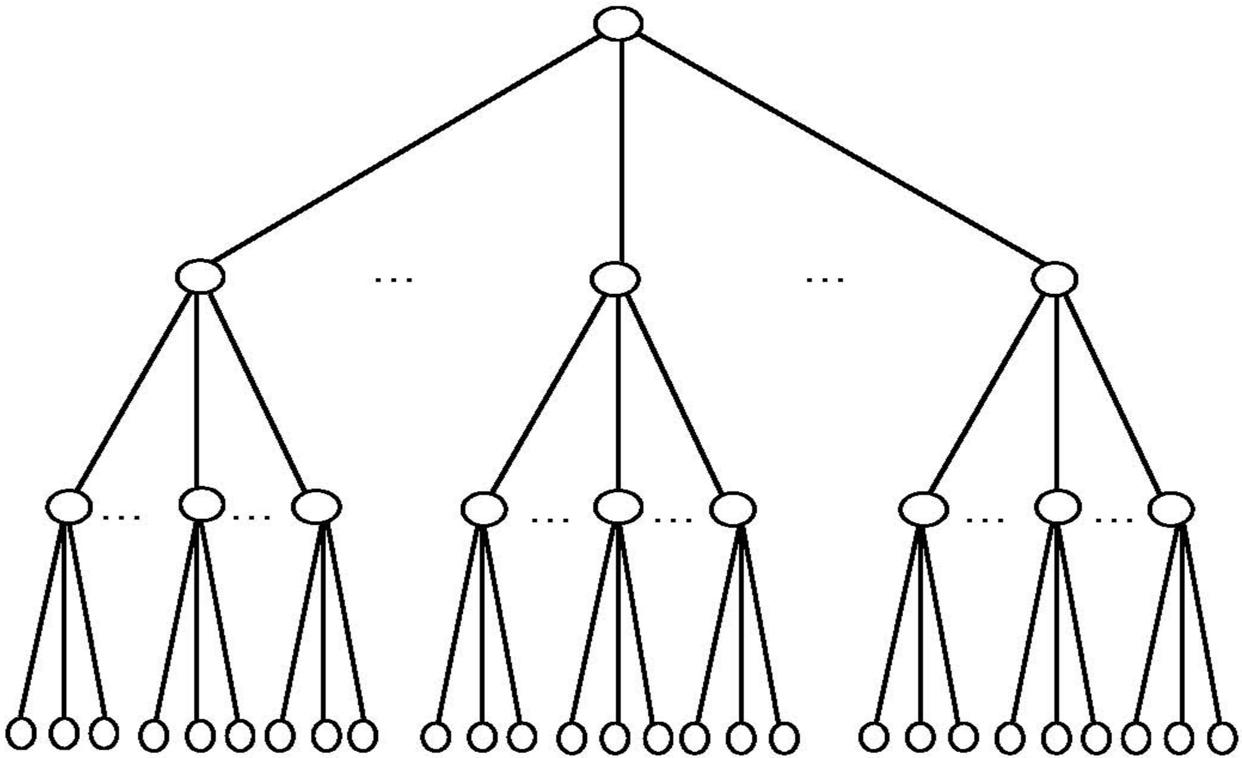
and where $V_{T+1} \equiv 0$

$V_t(\cdot, \xi_t)$ is piecewise linear and convex

SLP-T Assumptions for SDDP

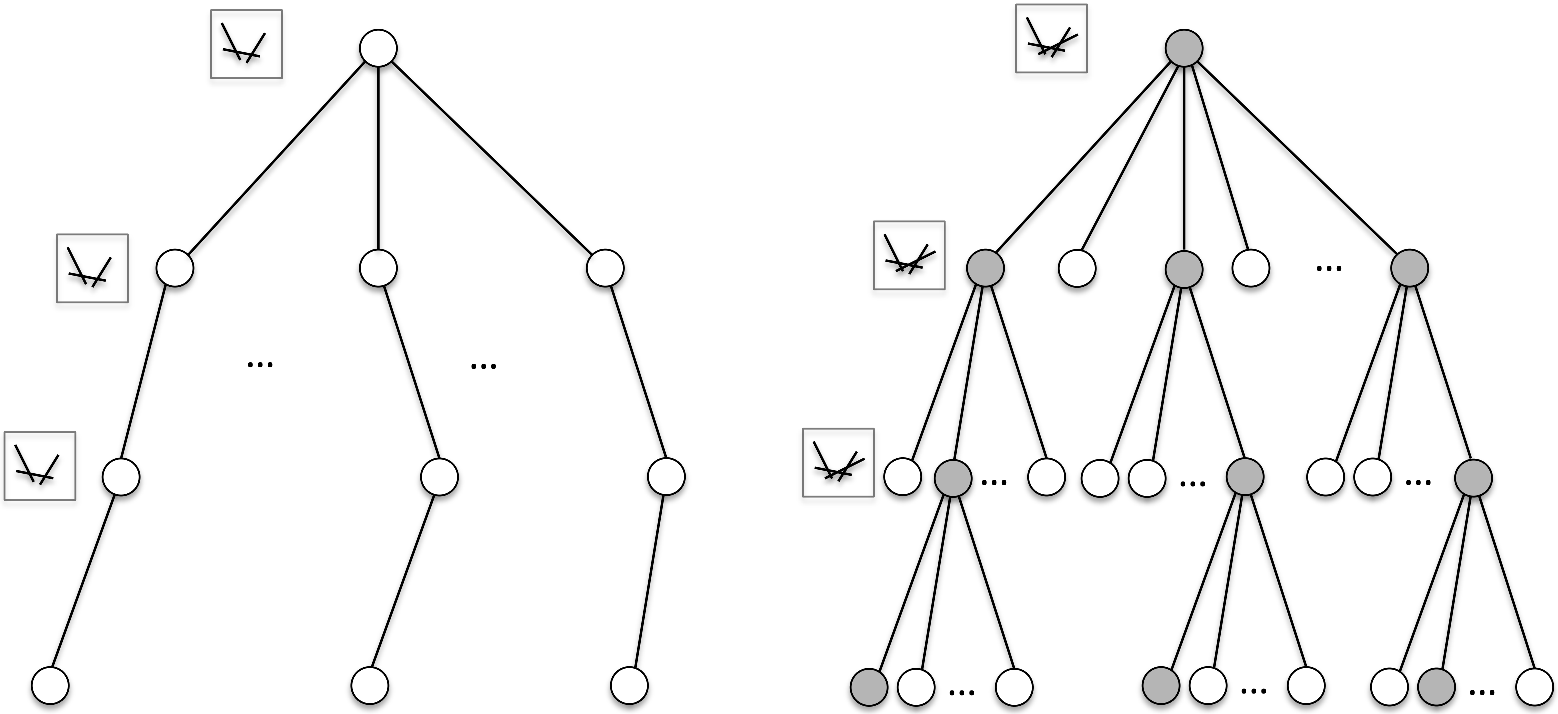
- Relatively complete recourse, finite optimal solution
- $\xi_t = (A_t, B_t, b_t, c_t)$ is inter-stage independent
- Or, (A_t, B_t, c_t) is inter-stage independent and b_t satisfies, e.g.,
  - $b_t = \Psi(b_{t-1}) + \varepsilon_t$ with ε_t inter-stage independent; or,
  - $b_t = \Psi(b_{t-1}) \cdot \varepsilon_t$ with ε_t inter-stage independent
- Sample space: $\Omega_t = \Sigma_2 \times \Sigma_3 \times \cdots \times \Sigma_t$ with $|\Sigma_t|$ modest
- T may be large

What Does "Solution" Mean?



A solution is a *policy*

SDDP



(a) Forward Pass (b) Backward Pass

SDDP Master Programs

$$
\begin{aligned}
\min_{x_t, \theta_t} \quad & c_t x_t + \theta_t \\
\text{s.t.} \quad & A_t x_t = B_t x_{t-1} + b_t \\
& -G_t^k x_t + \theta_t \geq g_t^k, k = 1, 2, \ldots, K \\
& x_t \geq 0
\end{aligned}
$$

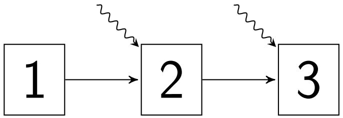
Partially Observable Multistage Stochastic Programming

Or, an alternative to DRO when you don't really know the distribution

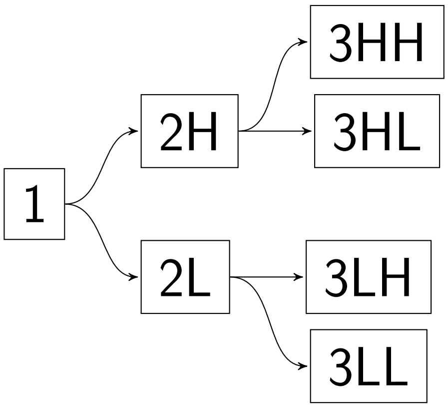
An apology: Not talking about Wasserstein-based DRO for SLP-T via an SDDP Algorithm (with Daniel Duque)

Policy Graphs (Dowson)

A policy graph for SLP-3 with inter-stage independence:

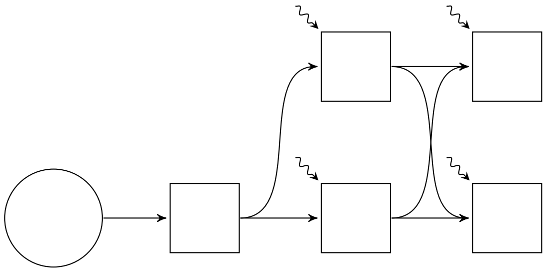


Unfolds to a scenario tree:

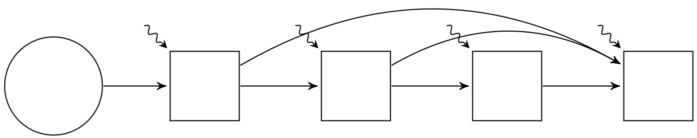


Policy Graphs

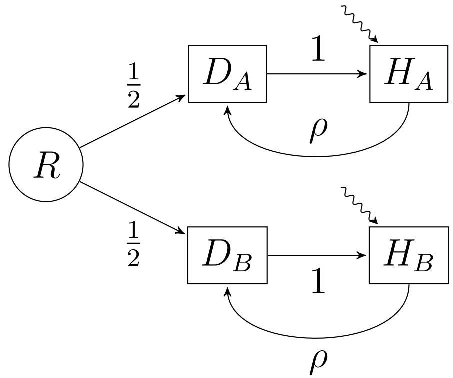
A Markov-switching model:



Random transitions:



Inventory Example



Demand model A: $\mathbb{P}(\omega = 1) = 0.2 \quad \mathbb{P}(\omega = 2) = 0.8$

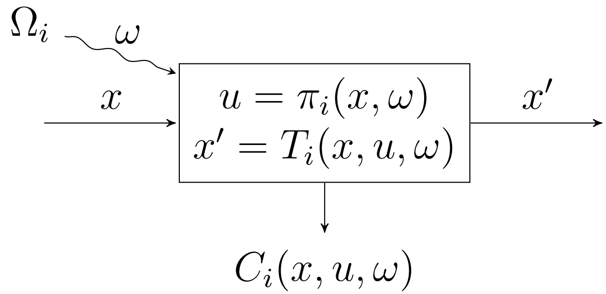
Demand model B: $\mathbb{P}(\omega = 1) = 0.8 \quad \mathbb{P}(\omega = 2) = 0.2$

$$
\begin{aligned}
D_i: \;\; D_i(x) = \min_{u, x' \geq 0} \;\; & u + \mathbb{E}_\omega[H_i(x', \omega)] \\
\text{s.t.} \;\; & x' = x + u
\end{aligned}
$$

$$
\begin{aligned}
H_i: \;\; H_i(x, \omega) = \min_{u, x' \geq 0} \;\; & 2u + x' + \rho D_i(x) \\
\text{s.t.} \;\; & x' = x + u - \omega
\end{aligned}
$$

Policy Graphs

Each node i:



A policy graph:

- $\mathcal{G} = (R, \mathcal{N}, \mathcal{E}, \Phi)$
- $\omega_j \in \Omega_j$: node-wise independent noise
- feasible controls: $u \in U_i(x, \omega)$
- transition function: $x' = T_i(x, u, \omega)$
- one-step cost function: $C_i(x, u, \omega)$

Policy Graphs

$$\min_{\pi} \mathbb{E}_{i \in R^+;\ \omega \in \Omega_i}[V_i(x_R, \omega)] \tag{1}$$

where

$$\begin{aligned} V_i(x, \omega) = \min_{u, \bar{x}, x'} \quad & C_i(\bar{x}, u, \omega) + \mathbb{E}_{j \in i^+;\ \varphi \in \Omega_j}[V_j(x', \varphi)] \\ \text{s.t.} \quad & \bar{x} = x \\ & u \in U_i(\bar{x}, \omega) \\ & x' = T_i(\bar{x}, u, \omega) \end{aligned} \tag{2}$$

Goal: Find $\pi_i(x, \omega)$ that solves (1) for each $i \in \mathcal{N}$, x, and ω

(A1) $\mathcal{N}$ is finite

(A2) Ω_i is finite and ω_i is node-wise independent $\forall i \in \mathcal{N}$

(A3) Excluding cost-to-go term, subproblem (2) is an LP

(A4) Subproblem (2) has finite optimal solution

(A5) Hit leaf node with probability 1 (or graph $\mathcal{G}$ is acyclic)

Policy Graphs with Partial Observability

Extend policy graph to:

$$\mathcal{G} = (R, \mathcal{N}, \mathcal{E}, \Phi, \mathcal{A})$$

where $\mathcal{A}$ partitions $\mathcal{N}$:

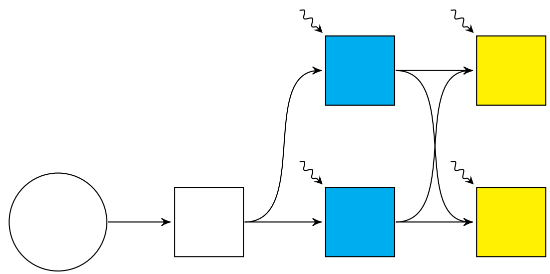
$$\bigcup_{A \in \mathcal{A}} A = \mathcal{N} \qquad A \cap A' = \emptyset, A \neq A'$$

We know the current ambiguity set, A, but not which node

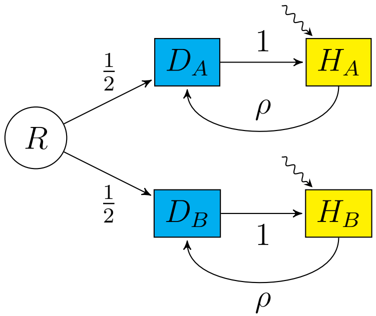
Full observability:

$$\mathcal{A} = \{\{i\} : \ i \in \mathcal{N}\}, \ \ \text{i.e., } |A| = 1$$

But, could have $|A| = 2$, where we know the stage but not the node



Updates to the Belief State



$\mathcal{A} = \{A_1, A_2\}$, with $A_1 = \{D_A, D_B\}$ and $A_2 = \{H_A, H_B\}$

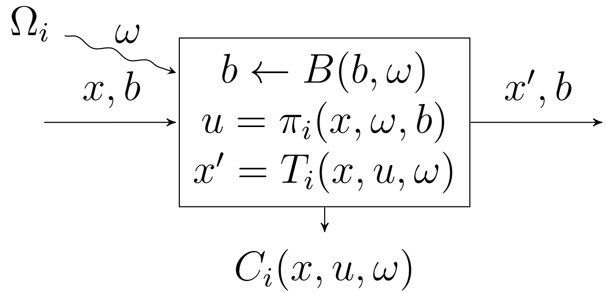
$$\mathbb{P}\{\text{Node} = k \,|\, \omega, A\} = \frac{\mathbf{1}_{k \in A} \cdot \mathbb{P}\{\omega \,|\, \text{Node} = k\}\mathbb{P}\{\text{Node} = k\}}{\mathbb{P}\{\omega\}}$$

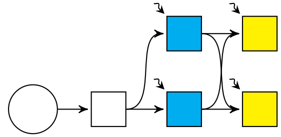
$$b_k \quad \leftarrow \quad \frac{\left[\mathbf{1}_{k \in A} \cdot \mathbb{P}(\omega \in \Omega_k)\right] \sum_{i \in \mathcal{N}} b_i \phi_{ik}}{\sum_{i \in \mathcal{N}} b_i \sum_{j \in A} \phi_{ij} \mathbb{P}(\omega \in \Omega_j)}$$

$$b \leftarrow B(b, \omega) \quad = \quad \frac{D_A^{\omega} \Phi^{\top} b}{\sum_{i \in \mathcal{N}} b_i \sum_{j \in A} \phi_{ij} \mathbb{P}(\omega \in \Omega_j)}$$

Policy Graphs with Partial Observability

Each node:





- All nodes in an ambiguity set have the same C_i, T_i, and U_i
- Children i^+, transition probabilities ϕ_{ij}, even Ω_i may differ

Policy Graphs with Partial Observability

$$\min_{\pi} \mathbb{E}_{i \in R^+;\ \omega \in \Omega_i}[V_i(x_R, B_i(b_R, \omega), \omega)] \tag{3}$$

where

$$\begin{aligned} V_i(x, b, \omega) = \min_{u, \bar{x}, x'} \ & C_i(\bar{x}, u, \omega) + \mathcal{V}(x', b) \\ \text{s.t.} \ & \bar{x} = x \\ & u \in U_i(\bar{x}, \omega) \\ & x' = T_i(\bar{x}, u, \omega) \end{aligned}$$

and where

$$\mathcal{V}(x', b) = \sum_{j \in \mathcal{N}} b_j \sum_{k \in \mathcal{N}} \phi_{jk} \sum_{\varphi \in \Omega_k} \mathbb{P}(\varphi \in \Omega_k) \cdot V_k(x', B_k(b, \varphi), \varphi)$$

Goal: Find $\pi_A(x, b, \omega)$ that solves (3) for each $A \in \mathcal{A}$, x, b, and ω

Saddle Property of Cost-to-go Function

$$\begin{aligned} V_i(x, b, \omega) = \min_{u, \bar{x}, x'} \quad & C_i(\bar{x}, u, \omega) + \mathcal{V}(x', b) \\ \text{s.t.} \quad & \bar{x} = x \\ & u \in U_i(\bar{x}, \omega) \\ & x' = T_i(\bar{x}, u, \omega) \end{aligned}$$

where

$$\mathcal{V}(x', b) = \sum_{j \in \mathcal{N}} b_j \sum_{k \in \mathcal{N}} \phi_{jk} \sum_{\varphi \in \Omega_k} \mathbb{P}(\varphi \in \Omega_k) \cdot V_k(x', B_k(b, \varphi), \varphi)$$

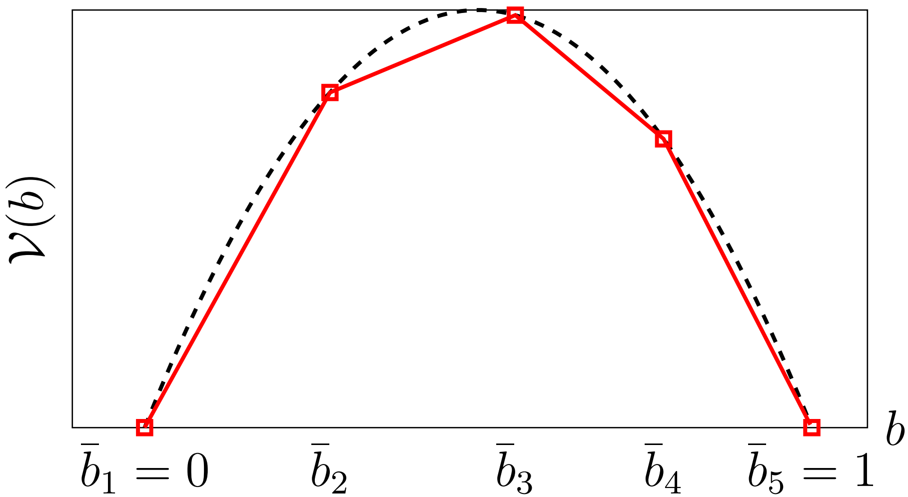
Assume (A1)-(A5) with $\mathcal{G}$ acyclic

Lemma 1. *Fix i, b, ω. Then, $V_i(x, b, \omega)$ is piecewise linear convex in x.*

Lemma 2. *Fix x'. Then, $\mathcal{V}(x', b)$ is piecewise linear concave in b.*

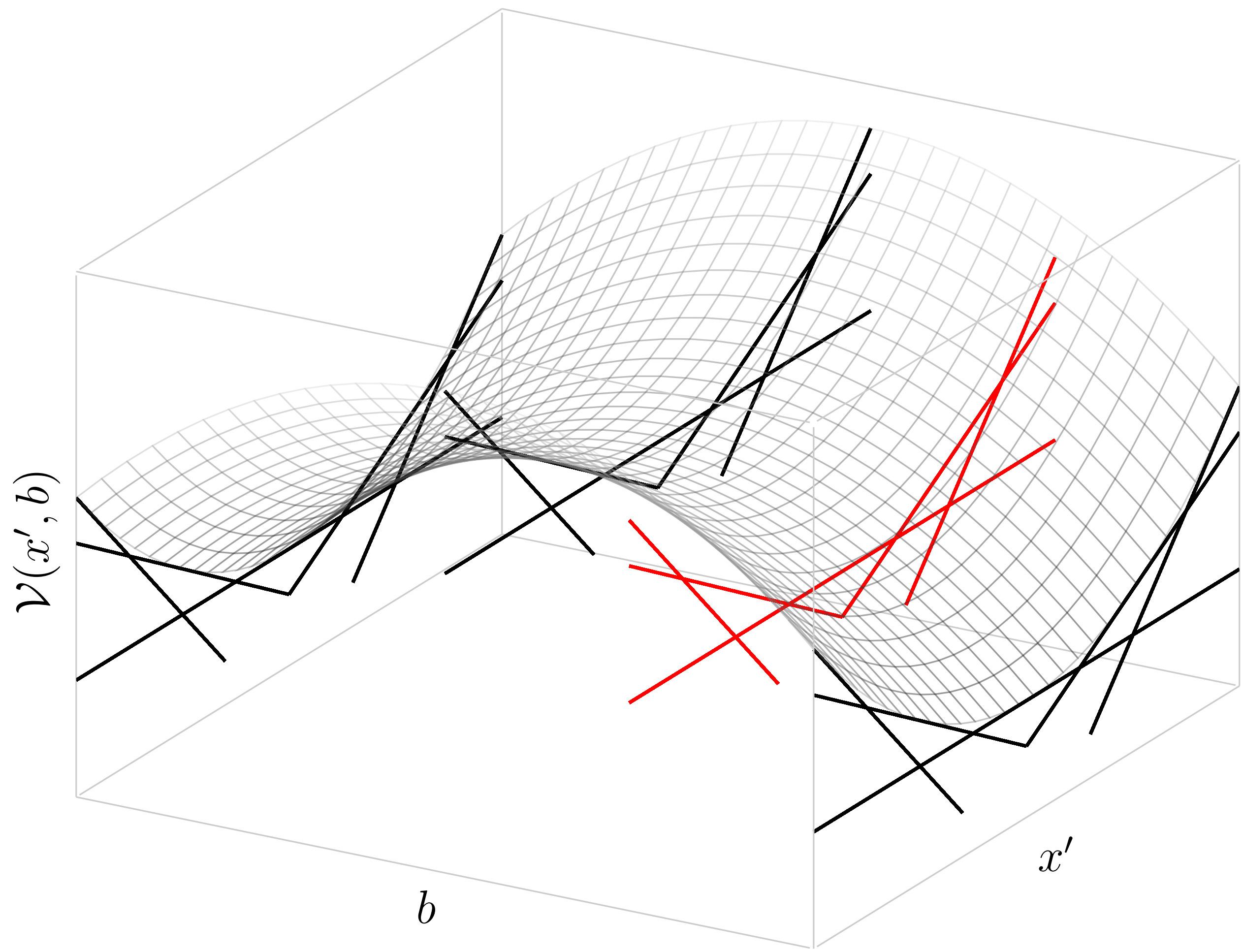
Theorem 1. *$\mathcal{V}(x', b)$ is a piecewise linear saddle function, which is convex in x' for fixed b and concave in b for fixed x'.*

Linear Interpolation: Towards an SDDP Algorithm

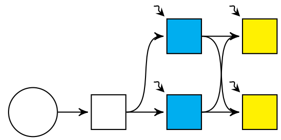


$$
\begin{aligned}
\mathcal{V}(b) = \max_{\gamma \geq 0} \quad & \sum_{k=1}^{K} \gamma_k \mathcal{V}(\bar{b}_k) \\
\text{s.t.} \quad & \sum_{k=1}^{K} \gamma_k = 1 \\
& \sum_{k=1}^{K} \gamma_k \bar{b}_k = b
\end{aligned}
$$

Saddle Function with Interpolated Cuts



Computing Cuts for What?



$$
\begin{aligned}
V_i(x, b, \omega) = \min_{u, \bar{x}, x'} \quad & C_i(\bar{x}, u, \omega) + \mathcal{V}_A(x', b) \\
\text{s.t.} \quad & \bar{x} = x \\
& u \in U_i(\bar{x}, \omega) \\
& x' = T_i(\bar{x}, u, \omega)
\end{aligned}
$$

where

$$
\mathcal{V}_A(x', b) = \sum_{j \in A} b_j \sum_{k \in j^+} \phi_{jk} \sum_{\varphi \in \Omega_k} \mathbb{P}(\varphi \in \Omega_k) \cdot V_k(x', B_k(b, \varphi), \varphi)
$$

SDDP Master Program

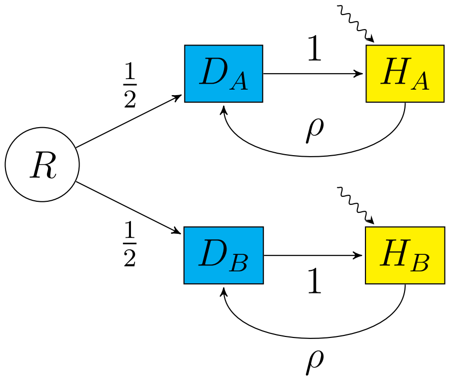
$$
\begin{aligned}
V_i^K(x,b,\omega) = \min_{u,\bar{x},x',\theta}\ \max_{\gamma\geq 0}\ \ & C_i(\bar{x},u,\omega) + \sum_{k=1}^{K} \gamma_k \theta_k \\
\text{s.t.}\ \ & \bar{x} = x && [\lambda] \\
& u \in U_i(\bar{x},\omega) \\
& x' = T_i(\bar{x},u,\omega) \\
& \sum_{k=1}^{K} \gamma_k b_k = b && [\mu] \\
& \sum_{k=1}^{K} \gamma_k = 1 && [\nu] \\
& \theta_k \geq G_k x' + g_k, \quad k = 1,\ldots,K
\end{aligned}
$$

SDDP Master Program

$$
\begin{aligned}
V_i^K(x,b,\omega) = \min_{u,\bar{x},x',\nu,\mu} \quad & C_i(\bar{x},u,\omega) + \mu^\top b + \nu \\
\text{s.t.} \quad & \bar{x} = x, \qquad\qquad\qquad\qquad\qquad [\lambda] \\
& u \in U_i(\bar{x},\omega) \\
& x' = T_i(\bar{x},u,\omega) \\
& \mu^\top b_k + \nu \geq G_k x' + g_k, \;\; k = 1,\ldots,K
\end{aligned}
$$

Theorem 2. *Assume (A1)-(A5) with $\mathcal{G}$ acyclic. Let the sample paths of the "obvious" SDDP algorithm be generated independently at each iteration. Then, the algorithm converges to an optimal policy almost surely in a finite number of iterations.*

Inventory Example



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$$
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\text{s.t.} \ & x' = x + u
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$$

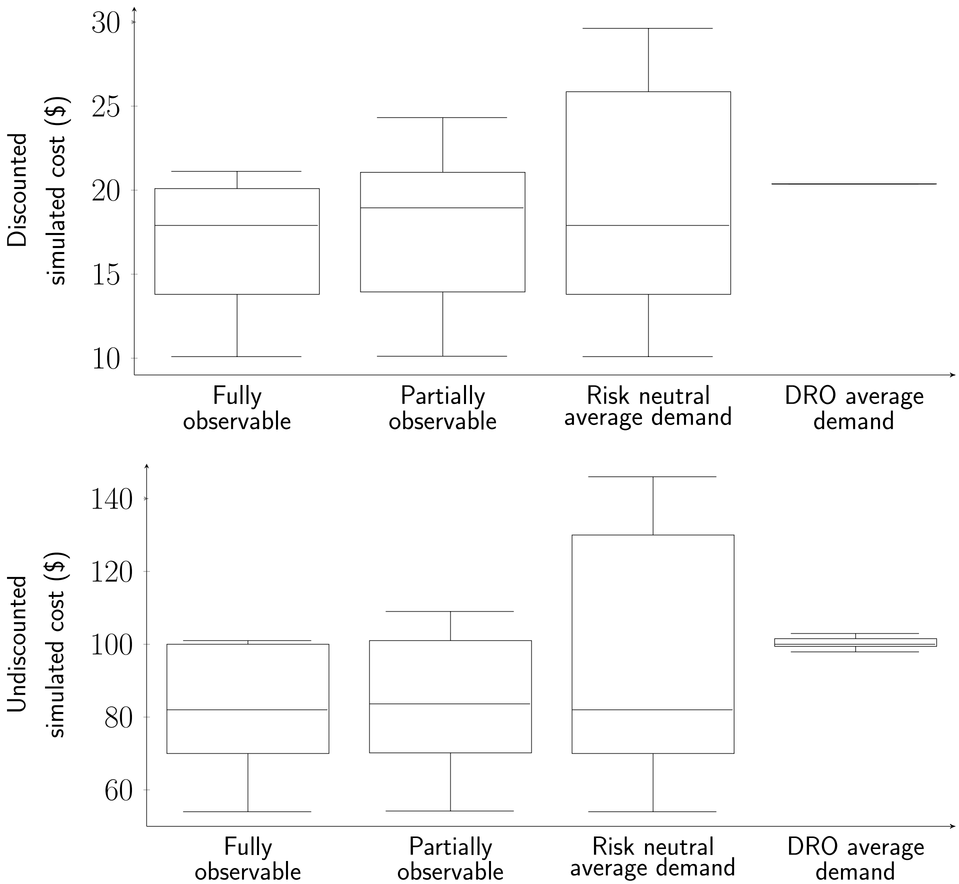
$$
\begin{aligned}
H_i : \ H_i(x, \omega) = \min_{u, x' \geq 0} \ & 2u + x' + \rho D_i(x) \\
\text{s.t.} \ & x' = x + u - \omega
\end{aligned}
$$

Inventory Example: Train Four Policies

1. *fully observable*: distribution known upon departing R
2. *partially observable*: ambiguity partition $\{D_A,\, D_B\}$, $\{H_A, H_B\}$
3. *risk-neutral average demand*: demand equally likely to be 1 or 2
4. *DRO average demand*: modified χ^2 method with radius 0.25

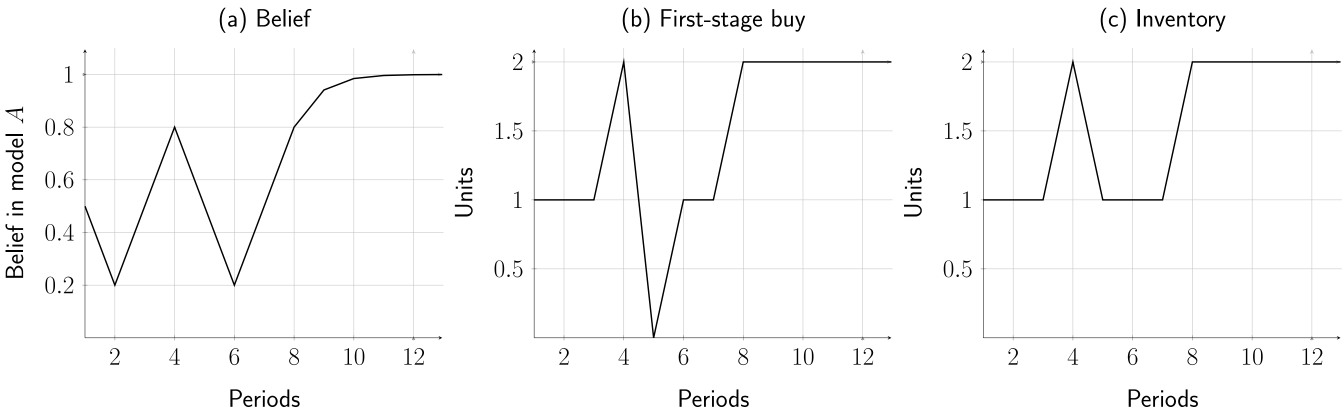
Inventory Example: Train Four Policies

- 2000 out-of-sample costs over 50 periods; quartiles; $\rho = 0.9$



Inventory Example

One Sample Path of the Partially Observable Policy



Concluding Thoughts

- Partially observable multistage stochastic programs
  - Saddle-cut SDDP algorithm
  - `SDDP.jl` (Dowson and Kapelevich)
- Related saddle-function work in stochastic programming
  - Baucke et al. (2018): risk measures
  - Downward et al. (2018): stage-wise dependent obj. coefficients
- Closely related ideas are well known in POMDPs
  - Contextual, multi-model, concurrent MDPs
  - We allow continuous state and action spaces via convexity
- Countably infinite LPs for cyclic case
- We did *not* handle decision-dependent learning
  - $b \leftarrow B(b, \omega)$ versus $b \leftarrow B(b, \omega, u)$

Concluding Thoughts

http://www.optimization-online.org/DB_HTML/2019/03/7141.html